"Anomaly" in $n = \infty$ Alday-Maldacena duality for wavy circle

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ABSTRACT: If the Alday-Maldacena version of string/gauge duality is formulated as an equivalence between double loop and area integrals *a la* arXiv: 0708.1625, then this pure geometric relation can be tested for various choices of n-side polygons. The simplest possibility arises at $n = \infty$, with polygon substituted by an arbitrary continuous curve. If the curve is a circle, the minimal surface problem is exactly solvable. If it infinitesimally deviates from a circle, then the duality relation can be studied by expanding in powers of a small parameter. In the first approximation the Nambu-Goto (NG) equations can be linearized, and the peculiar NG Laplacian $\Delta_{\mathrm{NG}} = \Delta_0 - \mathcal{D}^2 + \mathcal{D}$ plays the central role. Making use of explicit zero-modes of this operator (NG-harmonic functions), we investigate the geometric duality in the lowest orders for small deformations of arbitrary shape lying in the plane of the original circle. We find a surprisingly strong dependence of the minimal area on regularization procedure affecting "the boundary terms" in minimal area. If these terms are totally omitted, the remaining piece is regularization independent, but still differs by simple numerical factors like 4 from the double-loop integral which represents the BDS formula so that we stop short from the first non-trivial confirmation of the Alday-Maldacena duality. This confirms the earlier-found discrepancy for two parallel lines at $n = \infty$, but demonstrates that it actually affects only a finite number (out of infinitely many) of parameters in the functional dependence on the shape of the boundary, and the duality is only *slightly* violated, which allows one to call this violation an *anomaly*.

KEYWORDS: AdS-CFT Correspondence, Duality in Gauge Field Theories, Supersymmetric gauge theory, Nonperturbative Effects.

Contents

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1. Introduction

1.1 Alday-Maldacena duality

The Alday-Maldacena version [1] of the string-gauge duality [2] is one of the most spectacular new hypotheses of the last year and it naturally attracts an increasing attention [3]–[33]. We prefer to formulate it in a pure geometric form [12]:

Conjecture: an explicit regularization can be found such that for any polygon Π which is made from n light-like segments in Minkowski space R^4_{-+++}

$$D_\Pi \equiv \left(\oint \oint_\Pi \frac{d\vec{y}\, d\vec{y}\,'}{(\vec{y} - \vec{y}\,')^2} \right)\Bigg|_{\text{regularized}} \overset{?}{=} \left(\text{Minimal Area} \right)\Big|_{\text{regularized}} \equiv A_\Pi \qquad (1.1)$$

where A_Π is (regularized) area of a minimal surface in the bulk AdS_5 space with the metric

$$ds^2 = \frac{dr^2 + d\vec{y}^{\,2}}{r^2}, \quad d\vec{y}^{\,2} = -dy_0^2 + dy_1^2 + dy_2^2 + dy_3^2 \qquad (1.2)$$

bounded by the polygon Π which is located at the boundary (absolute) of the AdS_5 (at $r \to 0$).

1.2 Comments

The Alday-Maldacena duality is motivated by considerations of the planar ($N = \infty$) limit of $\mathcal{N} = 4$ SYM and combines a number of different hypotheses about the non-perturbative properties of this theory. Despite we are going to analyze (1.1) as a formal relation, without direct reference to its physical meaning, a few remarks are still necessary to clarify the possible subtleties of the problem. For more detailed presentation of our understanding of physical motivation behind (1.1) see [12, 21, 26, 29].

1. Π in (1.1) is a polygon in the *momentum* space, formed by n null momenta of external gluons. Therefore AdS_5 space at the r.h.s. of (1.1) is dual [34, 1] to the ordinary one in AdS/CFT correspondence [35]. Accordingly one needs to distinguish between conformal SO(4, 2) symmetries of the bulk and momentum AdS_5 spaces.

2. The l.h.s. of (1.1) looks like a logarithm of the average of an ordinary *Abelian* Wilson loop:

$$D_\Pi = \log \left\langle \exp\left\{ i \oint_\Pi A_\mu(\vec{y}) dy^\mu \right\} \right\rangle_{\text{regularized}} \tag{1.3}$$

eq. (1.1) should not be confused with another well-known conjecture,

$$A_\Pi \overset{?}{=} \log W_\Pi, \tag{1.4}$$

relating the r.h.s. of (1.1) to an average of the $N = 4$ SUSY Wilson loop

$$W_\Pi = \left\langle \text{Tr}\, P \exp\left\{ i \oint_\Pi \left(A_\mu(\vec{y}) dy^\mu + \phi\, dl \right) \right\} \right\rangle_{\text{regularized}} \tag{1.5}$$

involving non-Abelian vector fields and scalars and non-trivial multi-loop diagrams.

3. The l.h.s. of (1.1) is an identical, though non-trivial, reformulation [1, 5, 6] of the celebrated BDS conjecture [36], stating that the n-gluon MHV amplitude in $\mathcal{N} = 4$ SUSY YM in the planar limit is *exactly* equal to the exponential of the one-loop result, which is in turn reduced to contribution of the "2me" box diagrams and explicitly expressed through dilogarithm functions [37, 38].

4. The r.h.s. of (1.1) can be considered as a version of the Gross-Mende conjecture [39] that the high-energy asymptotics of stringy scattering amplitudes are given by exponentiated minimal areas in the relevant bulk spaces with appropriate boundary conditions. Within the $\mathcal{N} = 4$ SUSY context, one can assume that the statement is true for all values of external momenta, not obligatory large, while the ADS/CFT conjecture [35] identifies the relevant bulk space in this case as $AdS_5 \times S^5$.

1.3 Current status of the Alday-Maldacena duality

The status is somewhat controversial.

All reliable evidence in support of (1.1) is at $n = 4$ [1] and sometime at $n = 5$ [21, 16, 28]. Unfortunately, this evidence is not decisive, because at $n = 4, 5$ explicit expressions are fully determined by the anomalous Ward identities [31], associated with the global conformal invariance of the problem [5, 16, 28, 30]. For $n \geq 6$ this symmetry is too small to unambiguously constrain the answer, but in this case there is still no clear way to explicitly evaluate the r.h.s. of (1.1). This Plateau minimal-surface problem is considered unresolvable (in any explicit form) in flat spaces. If (1.1) was true, this would imply that the situation is drastically different in AdS space, since the l.h.s. is an absolutely explicit expression: the AdS Plateau problem would be *exactly solvable*, and this is what makes the Alday-Maldacena hypothesis so interesting and significant far beyond $\mathcal{N} = 4$ SUSY studies. Attempts to solve the AdS Plateau problem are described in [26, 29], but they are still far from being conclusive.

Meanwhile, the counter-arguments *against* (1.1) are mounting. Already known ones can be divided into three categories.

Counter-arguments of the first type argue that the BDS conjecture, which is behind the l.h.s. of (1.1), contradicts some other physically-expected properties of the scattering amplitudes for $n \geq 6$, like Regge behavior [32].

The second type of counter-arguments [30] is based on results of higher-loop calculations of non-Abelian Wilson average W_Π. The claim is that $D_\Pi \neq \log W_\Pi$, so that (1.1) comes in contradiction with the usual belief that $A_\Pi = \log W_\Pi$. This belief is just supported once again by [40, 41].

The third type [23, 20] comes from attempts to evaluate A_Π for some special polygons Π, when the AdS Plateau problem is simplified. While in [20] the boundary conditions are considered which seem to be inconsistent with the simplest BDS conjecture (additional restrictions on virtual momenta in the loops are imposed), the discrepancy found in [23] can be eliminated only by an ugly change of regularization, what signals about a real problem.

All these difficulties look very serious and seem to distract people from the Alday-Maldacena hypothesis, at least, in its simplest form (1.1). However, the above-mentioned counter-arguments have a common drawback: they are too special to show any way out, they can serve only to rule out formula (1.1), but can not explain how and why it should be modified. Thus, one needs at this moment is a considerable extension of the above counter-examples, taking them from particular selected points in the infinite-dimensional "moduli space" of all relevant polygons Π to at least some vicinities of those: this can help to get rid of regularization ambiguities (provided there is only a finite number of possible counterterms) or to formulate explicit requirements to infinite-parametric regularization schemes (if one is going to look for a resolution of emerging problems this way).

1.4 The goal of this paper: a perturbative analysis of the smooth $n = \infty$ limit

In this paper we are going to elaborate on the so far most constructive counter-example to (1.1): the one found in [23] for a special rectangular configuration at $n = \infty$. The specifics of this "smooth $n = \infty$ limit", see s.2.8 of [26], is that the Plateau problem can be reduced from AdS_5 to Euclidean AdS_3 lying at $y_0 = y_3 = 0$, and Π in (1.1) becomes an *arbitrary* curve in the plane of the complex variable $z = y_1 + iy_2$. One can apply the methods, developed in [26, 29], to solve the Plateau problem, at least, in the from of power series in the deviations from some exactly-solvable examples where the role of Π is played by two parallel lines or a circle. This kind of slightly deviating boundary conditions was called "wavy" in [42] (see also [43]), and, in these terms, we are going to address the problem of "the wavy circle". The long-rectangular (actually, the two-parallel-lines) example of [23] would correspond to a circle of infinite radius, however, this large-radius limit is somewhat singular and "wavy rectangular" requires separate consideration, which is straightforward, but left beyond the scope of the present paper.[1] In this way we obtain the l.h.s. and the r.h.s. of (1.1) for an infinitely-parametric family of wavy curves Π and thus obtain a significantly wider information than in the previous considerations.

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[1]One could of course use the results of [43, 42], but since the σ-model action was used there instead of the Nambu-Goto one, there are additional sources of complications.

Our result is somewhat surprising: we confirm that (1.1) is not true, at least, in the most naive regularization prescription. However, even for this prescription the two sides of (1.1) are very similar. Still, they are different, moreover, their global conformal properties do not coincide. At the same time, we observe an unexpectedly strong dependence on the choice of regularization prescription, what makes the hypothesis formulated in s.1.1, much more difficult to overturn.

1.5 The main result of this paper

Our attempt to confirm relation (1.1) for a continuous curve $\Pi = \bar{\Pi}$, which is an infinitesimal deformation of a unit circle in the complex z-plane, $z = y_1 + iy_2$, with $y_0 = y_3 = 0$, **fails**, but in an interesting and puzzling way: the two sides of (1.1) are different, but only *slightly* different.

Namely, if Π is an image of the unit circle $|\zeta|^2 = 1$ under the conformal map $z = H(\zeta) = \zeta + \sum_{k=0}^{\infty} h_k \zeta^k$, then

$$\frac{D_\Pi}{2\pi} = \frac{L}{\lambda} - 2\pi - 4\pi\left[Q_\Pi^{(2)} - Q_\Pi^{(3,1)} - Q_\Pi^{(3,2)}\right] + 4\pi Q_\Pi^{(4)} + O(h^5), \quad (1.6)$$

$$\frac{A_\Pi}{2\pi} = \frac{L}{4\mu} - 1 - \frac{3}{2}\left[Q_\Pi^{(2)} - Q_\Pi^{(3,1)} - 4Q_\Pi^{(3,2)}\right] + O(h^4) \quad (1.7)$$

We see the discrepancy between these two expressions: first the coefficients in front of the brackets differ by a factor of $\kappa_\circ = \frac{8\pi}{3}$, second, one of the structures in the brackets in A_Π differs from those in D_Π by a mysterious integer factor 4. Thus, only few of infinitely many coefficients in h-expansions are different, still the difference exists even if regularizations are matched, $\kappa_\circ \lambda = 4\mu$ and nonphysical constants 2π and 1 are omitted.

Moreover, one could even think that the overall coefficient $\kappa_\circ$ is not a problem at all. However, it is, if one assumes this coefficient is completely independent on the shape of Π. Indeed, in the quadrilateral $n = 4$ example [1, 12, 22][2] $\kappa_\square = 8$ and, therefore, $\kappa_\circ = \frac{8\pi}{3} = \frac{\pi}{3}\kappa_\square$. Still, one can imagine a simple dependence of this coefficient only on the number of corners of Π to reproduce this overall difference $\frac{\pi}{3}$.

In these formulas $Q^{(p,q)}$ are certain structures of the order h^p:

$$Q_\Pi^{(2)} = \sum_{k=0}^{\infty} B_k |h_k|^2,$$

$$B_k = \frac{k(k-1)(k-2)}{6}, \quad (1.8)$$

$$Q_\Pi^{(3)} = Q_\Pi^{(3,1)} + Q_\Pi^{(3,2)} = \frac{1}{2}\sum_{i,j=0}^{\infty} C_{ij}\left(h_i h_j \bar{h}_{i+j-1} + \bar{h}_i \bar{h}_j h_{i+j-1}\right),$$

$$C_{ij} = \frac{ij}{6}\left(i^2 + 3ij + j^2 - 6i - 6j + 7\right) \quad (1.9)$$

$$Q_\Pi^{(3,2)} = \frac{1}{2}\sum_{i=0}^{\infty} C_{ii}\left(h_i^2 \bar{h}_{2i-1} + \bar{h}_i^2 h_{2i-1}\right), \quad (1.10)$$

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[2]In this example, the finite piece of the double loop integral is $-2(\log s/t)^2$ (see (2.16)-(2.17) and (2.13) in [12]), while that of the minimal area is $-1/4(\log s/t)^2$ [1, 22], compare with $-1/2(\log s/t)^2$ in the σ-model case [12, (4.26)].

while $Q_\Pi^{(3,1)}$ is the sum of off-diagonal terms,

$$Q_\Pi^{(3,1)} = \sum_{i<j}^\infty C_{ij}\Big(h_i h_j \bar{h}_{i+j-1} + \bar{h}_i \bar{h}_j h_{i+j-1}\Big) \tag{1.11}$$

Different coefficients in front of $Q_\Pi^{(3,1)}$ and $Q_\Pi^{(3,2)}$ in A_Π imply that the tensor $C_{ij}^{(A)} = 3Q_{ij}^{(3,1)} + 12Q_{ij}^{(3,2)}$, which would play the role of $C_{ij} = C_{ij}^{(D)} = Q_{ij}^{(3,1)} + Q_{ij}^{(3,2)}$ in (1.7), is *not* just a polynomial in the indices i, j.

$$Q_\Pi^{(4)} = (h_1^2 + \bar{h}_1^2)Q_\Pi^{(2)} + \frac{1}{4}\sum_{\substack{i,j,k,l=0 \\ i+j=k+l}}^\infty U_{ij;kl} h_i h_j \bar{h}_k \bar{h}_l$$

$$+ \frac{1}{6}\sum_{i,j,k=0}^\infty V_{ijk}\Big(h_i h_j h_k \bar{h}_{i+j+k-2} + \bar{h}_i \bar{h}_j \bar{h}_k h_{i+j+k-2}\Big),$$

$$U_{ij;kl} = \delta_{i+j,k+l}\bigg(kC_{ij} - \frac{1}{6}(i+j)(k+1)k(k-1)(k-2)$$

$$+ \frac{1}{10}(k+2)(k+1)k(k-1)(k-2)\bigg), \quad \text{for } k \le i,j,$$

$$V_{ijk} = \frac{ijk}{3}\Big(i^2 + j^2 + k^2 + 3(ij + jk + ik) - 9(i+j+k) + 15\Big) \tag{1.12}$$

The complete expression for $U_{ij;kl}$ is restored by the symmetry under the permutation $(i,j) \leftrightarrow (k,l)$. Note that the naive continuation of formula (1.12) to the whole region of indices leads to the non-symmetric $U_{ij;kl}$. Therefore, in this case already $U_{ij;kl} = U_{ij;kl}^{(D)}$ is not a polynomial of indices i, j, k, l. Terms of the order h^4 in A_Π still need to be calculated, presumably, they will also be made from the same coefficients $U_{ij;kl}$ and V_{ijk} but with *a few* extra overall coefficients as it happens to the h^2 and h^3 terms.

At least, the $\bar{h}$-linear terms in D_Π with all possible powers of h can be summed up to give

$$\oint (\bar{z} - \bar{\zeta})\mathcal{S}_\zeta\{z\}\zeta^2 d\zeta = \oint \bar{h}(\bar{\zeta})\left[\frac{h'''(\zeta)}{1 + h'(\zeta)} - \frac{3}{2}\left(\frac{h''(\zeta)}{1 + h'(\zeta)}\right)^2\right]\zeta^2 d\zeta \tag{1.13}$$

$$= \sum_{p=1}^\infty (-)^{p-1}\sum_{i_1,\dots,i_p=0}^\infty \frac{i_1 \dots i_p}{6p}\bigg(\sum_{a=1}^p i_a^2 + 3\sum_{a<b}^p i_a i_b$$

$$- 3p\sum_{a=1}^p i_a + \frac{p(3p+1)}{2}\bigg)h_{1_1}\dots h_{i_p}\bar{h}_{i_1+\dots+i_p+1-p}$$

The above coefficients A, C, V arise in particular terms of this formula, with $p = 1, 2, 3$ respectively and

$$\mathcal{S}_\zeta\{z\} = \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2 = -\frac{1}{z'^2}\mathcal{S}_z\{\zeta\}, \qquad \mathcal{S}_\zeta\{z\}\frac{d\zeta}{\sqrt{dz/d\zeta}} = -\mathcal{S}_z\{\zeta\}\frac{dz}{\sqrt{d\zeta/dz}} \tag{1.14}$$

is the Schwarzian derivative, which vanishes identically for rational transformations $z = \zeta + h(\zeta) = \frac{a\zeta+b}{c\zeta+d}$. Of course, there is a complex conjugate contribution which is linear in

h and sums up all possible powers of $\bar{h}$. It is unclear if a similar *local* expression can be found for all other terms $h^p\bar{h}^q$ in D_Π with both $p, q \geq 2$. Even less clear is the situation with A_Π.

We use λ and μ regularizations at the l.h.s. and at the r.h.s. of (1.1) respectively:

$$D_\Pi = \left(\oint_\Pi \oint \frac{d\vec{y}d\vec{y}'}{(\vec{y}-\vec{y}')^2 + \lambda^2} \right) \stackrel{\lambda=\mu}{\cong} \int \frac{\sqrt{|\partial H|^2(|\partial H|^2 + |\partial r|^2)}}{r^2 + \mu^2} d^2\zeta = A_\Pi \quad (1.15)$$

Divergent contributions are proportional to the length of the curve Π,

$$\frac{L}{2\pi} = 1 + \frac{1}{2}(h_1 + \bar{h}_1) - \frac{1}{8}(h_1^2 + \bar{h}_1^2) + \frac{1}{4}\sum_{k=1}^\infty \frac{k^2|h_k|^2}{|1+h_1|^2} - \frac{1}{4}h_1\bar{h}_1\left(h_1 + \bar{h}_1\right) + \frac{1}{16}(h_1^3 + \bar{h}_1^3) -$$

$$- \frac{1}{16}\sum_{k,l=2}^\infty kl(k+l-1)\left[\frac{h_k h_l \bar{h}_{k+l-1}}{(1+h_1)^2(1+\bar{h}_1)} + \frac{\bar{h}_k \bar{h}_l h_{k+l-1}}{(1+\bar{h}_1)^2(1+h_1)} \right] + O(h^4)$$

$$= \sqrt{1+h_1}\sqrt{1+\bar{h}_1}\left(1 + \frac{1}{4}\sum_{k=2}\frac{k^2|h_k|^2}{|1+h_1|^2} \right.$$

$$\left. - \frac{1}{16}\sum_{k,l=2}^\infty kl(k+l-1)\left[\frac{h_k h_l \bar{h}_{k+l-1}}{(1+h_1)|1+h_1|^2} + \frac{\bar{h}_k \bar{h}_l h_{k+l-1}}{(1+\bar{h}_1)|1+h_1|^2} \right] + \ldots \right) \quad (1.16)$$

To summarize, the functional dependencies on *arbitrary* shape of the curve Π in D_Π and A_Π are almost the same, but some overall coefficients are different, moreover, the number of different coefficients can grow with the order of h-corrections.

It is unclear if this difference can be somehow absorbed into the change of regularization prescriptions. Moreover, if instead of μ-regularization at the r.h.s. of (1.15), one cuts the area integral at $|\zeta| = 1 - c$, the answer for A_Π changes drastically, leaving no observable similarity to D_Π. Worse than that, while the IR-finite part of D_Π is invariant w.r.t. the projective transformations $\delta z = \epsilon_- + \epsilon_0 z + \epsilon_+ z^2$, i.e. is annihilated by the three SL(2) generators

$$\hat{J}_- = \frac{\partial}{\partial h_0},$$

$$\hat{J}_0 = \frac{\partial}{\partial h_1} + \sum_{k=0}^\infty h_k \frac{\partial}{\partial h_k},$$

$$\hat{J}_+ = \frac{\partial}{\partial h_2} + 2\sum_{k=0}^\infty h_k \frac{\partial}{\partial h_{k+1}} + \sum_{k,l=0}^\infty h_k h_l \frac{\partial}{\partial h_{k+l}}, \quad (1.17)$$

this is not true for the IR-finite part of A_Π (actually in the h^3 approximation $\hat{J}_+ A_\Pi^{\text{finite}} \neq 0$ only because of a wrong coefficient in front of a single term $h_2^2\bar{h}_3$, but there can be more such bad terms when the power of h increases). In fact, this does not immediately contradict the conformal invariance of A_Π, proved in [31]: the conformal symmetry of [31] acts on A_Π in a more sophisticated way than (1.17).

We refrain from making far-going conclusions from these surprising results before they are independently checked. In case if they are confirmed, they need and can be straightforwardly extended in two obvious directions: to higher orders in h-expansion and to "wavy

lines". This can help to better understand the structure of the difference between A_Π and D_Π and hopefully find a simple formulation of **the anomaly in the Alday-Maldacena duality** (1.1). Of course, this anomaly should be also extended to finite-n polygons Π. We emphasize that the apparent similarity between (1.7) and (1.6) does not allow one to simply reject (1.1) (say, by claiming the failure of the BDS conjecture), the relation looks too close to truth to be simply ignored: one should rather search for overlooked *corrections*, which can be responsible for the *small* discrepancy between the l.h.s. and the r.h.s. of (1.1).

1.6 Plan of the paper

Below in this paper we provide a rather detailed derivation of formulas (1.6) in section 3 and (1.7) in section 4, ending up with two simple MAPLE programs which can be used for double-check and generalizations. These derivations are preceded in section 2 by a plan of such calculation, commenting on various semi-technical issues, which can be useful for further generalizations. Then, there is a brief discussion of global conformal symmetry in s.5. Finally, the four appendices contain the derivation of formula for the circumference of the wavy circle and other local counterterms in terms of parameters of the conformal map, an alternative calculation of D_Π using a different regularization, a discussion of another, rectangular example that allows one to test formula (1.1), [23] and two MAPLE programs that allow one to calculate A_Π and D_Π.

2. Wavy circle: the scheme of calculations

2.1 NG equation for $y_0 = 0$

NG action with $y_0 = y_3 = 0$ is quite simple,

$$\int \frac{\sqrt{1 + (\partial_1 r)^2 + (\partial_2 r)^2}}{r^2} \, dy_1 dy_2 \tag{2.1}$$

and equation of motion is:

$$r\partial^2 r + 2(\partial r)^2 + 2 + r\partial_i r \partial_j r \left(\delta_{ij}\partial^2 r - \partial^2_{ij} r \right) = 0 \tag{2.2}$$

or

$$r\partial_i r \partial_j r \partial^2_{ij} r = \left(1 + (\partial r)^2 \right)\left(2 + r\partial^2 r \right) \tag{2.3}$$

There are a few exactly solvable examples that satisfy both the NG equation (2.2) and the boundary condition $r(y^2 = 1) = 0$. Unfortunately, they do not possess free parameters that can be used to actually compare the l.h.s. and the r.h.s. of (1.1). In particular, the surface

$$r^2 = R^2 - y^2 = R^2 - y_1^2 - y_2^2 = R^2 - z\bar{z} \tag{2.4}$$

is a solution to (2.2). It, indeed, provides a minimum of the regularized action. Later on, we put $R = 1$. In fact, changing R is the zero-mode generated by the coefficient h_1 of the conformal map. In fact, as illustrated by (1.16), h_1 enters all formulae in a special way, different from all other h_k. Therefore, for the sake of simplicity, we always put $h_1 = \bar{h}_1 = 0$ and restore non-vanishing h_1 and $\bar{h}_1$ only in s.5.

2.2 Wavy circle: area calculation

Here we consider an arbitrary infinitesimally deformed circle and describe how to calculate its regularized minimal area in the first non-trivial — quadratic — order in deformation parameters.

To this end, we need to resolve the following problems:

- Choose an adequate parametrization of the deformation. We do this by considering the conformal map $z = H(\zeta)$ of interior of the unit circle in the complex ζ-plane into the domain bounded by the deformed curve Π in the complex z-plane. The map is an infinitesimal deformation of the unit map, $H(\zeta) = \zeta + h(\zeta)$ and

$$h(\zeta) = \sum_{k=0}^{\infty} h_k \zeta^k \tag{2.5}$$

is a small function-valued parameter.

- Find the shape of the minimal surface $r^2(z, \bar{z}) = 1 - \zeta\bar{\zeta} + a(\zeta, \bar{\zeta})$ by solving the NG equation for $a(\zeta, \bar{\zeta})$ and imposing the boundary condition

$$a\left(e^{i\phi}, e^{-i\phi}\right) = 0 \tag{2.6}$$

For $h \neq 0$ vanishing everywhere $a = 0$ is not a solution, and one needs to calculate a up to the second order in h. The relevant form of the NG equation in this case is

$$\Delta_{\rm NG}\left(a + u(h)\right) = O(a^2, ah, h^2) \tag{2.7}$$

where $\Delta_{\rm NG}$ is a linear differential operator (already found in [29])

$$\Delta_{\rm NG} = \Delta_0 - \mathcal{D}^2 + \mathcal{D} = 4\partial\bar{\partial} - \bar{z}^2\bar{\partial}^2 - 2z\bar{z}\partial\bar{\partial} - z^2\partial^2 \tag{2.8}$$

expressed through the ordinary Laplace and dilatation operators $\Delta_0 = \partial_1^2 + \partial_2^2$ and $\mathcal{D} = y_1\partial_1 + y_2\partial_2$, and

$$u(h) = 2\zeta\bar{\zeta}\sum_{k=1}^{\infty} \mathrm{Re}\left(h_k\zeta^{k-1}\right) \tag{2.9}$$

is linear in h.

   A. As a first step towards solving (2.7) we can put $h = 0$ and neglect the quadratic term $\mathcal{Q}(a)$, i.e. consider the equation

$$\Delta_{\rm NG}(a) = 0 \tag{2.10}$$

Its generic solution was found in [26, 29] in the form

$$a(\zeta, \bar{\zeta}) = 2\sum_{k=0}^{\infty} \mathrm{Re}\left(a_k\zeta^k\right)F_k(\zeta\bar{\zeta}) \tag{2.11}$$

where

$$F_k(x) = \frac{(1 + k\sqrt{1-x})(1 - \sqrt{1-x})^k}{x^k} \sim {}_2F_1\left(\frac{k}{2}, \frac{k-1}{2}; k+1; x\right) \quad (2.12)$$

are specific hypergeometric functions expressed through the Legendre (spherical) functions $Q_{k-1/2}^{-3/2}$

$${}_2F_1\left(\frac{k}{2}, \frac{k-1}{2}; k+1; x\right) = 2^k k(k-1)i\sqrt{\frac{2}{\pi}}\left(\frac{1-x}{x}\right)^{\frac{3}{4}} x^{\frac{1-k}{2}} Q_{k-1/2}^{-3/2}\left(\frac{1}{\sqrt{x}}\right) \quad (2.13)$$

We normalize $F_k(x)$ by the condition

$$F_k(1) = 1, \quad (2.14)$$

i.e. divide the hypergeometric series at the r.h.s. of (2.12) by their values at $x = 1$,

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad (2.15)$$
$$\text{Re } c > \text{Re } b > 0, \quad \text{Re }(c-a-b) > 0$$

$${}_2F_1\left(\frac{k}{2}, \frac{k-1}{2}; k+1; 1\right) = \frac{2^k}{k+1} \quad (2.16)$$

In particular,

$$\begin{aligned}
F_0(x) &= 1, \\
F_1(x) &= 1, \\
F_2(x) &= \frac{-2 + 3x + 2(1-x)^{3/2}}{x^2}, \\
F_3(x) &= \frac{-8 + 12x - 3x^2 + 8(1-x)^{3/2}}{x^3}, \\
F_4(x) &= \frac{-24 + 40x - 15x^2 + 8(6-x)(1-x)^{3/2}}{x^4}, \\
&\cdots
\end{aligned} \quad (2.17)$$

In the vicinity of $x = 1$ these F_k behave as follows:

$$\text{at} \quad x = 1 - c^2 \qquad F_k = 1 - \frac{k(k-1)}{2}c^2 - \frac{k(k^2-1)}{3}c^3 + O(c^4) \quad (2.18)$$

Of course, for $h = 0$ the boundary condition (2.6) implies that in (2.11) all $a_k = 0$.

B. Since in neglect of its r.h.s. (2.7) differs from (2.10) only by a shift of a, we can use the same result (2.11) for $a + u(h)$. Moreover, the explicit form (2.9)

of the shift $u(h)$ is very simple, so that one can easily impose the boundary conditions (2.6)

$$a(\zeta, \bar{\zeta}) = 2 \sum_{k=1}^{\infty} \text{Re}\Big(h_k \zeta^{k-1}\Big) A_k(\zeta\bar{\zeta}) + O(h^2),$$
$$A_k(x) = F_{k-1}(x) - x \tag{2.19}$$

and, according to (2.18),

$$\text{at} \quad x = 1 - c^2 \qquad A_k = -\frac{k(k-3)}{2}c^2 + \frac{k(k-1)(k-2)}{3}c^3 + O(c^4),$$
$$A_k' = \frac{k(k-3)}{2} - \frac{k(k-1)(k-2)}{2}c + O(c^2) \tag{2.20}$$

- Evaluate (regularized) effective action up to the h^2-terms. It diverges and we regularize it. It can be done in many different ways, here we use the two most naive possibilities which are, however, representative enough to illustrate the typical features. As we shall see, the result drastically depends on the choice of regularization.

According to [1], the regularization procedure implies modifying the action but using the old solution (which is, definitely, a somewhat controversial prescription).

According to [44] the most appropriate way to regularize AdS quantities is to make a shift away from the boundary at $r = 0$ to $r = \epsilon$: dependence of the bulk action on the shift is the counterpart of renormalization group for the boundary theory.

The question in our case is where we impose the vanishing boundary conditions: on the boundary or on the shifted boundary?

Another question is what kind of shift we should perform: it can be of an arbitrary shape and the corresponding renormalization group is in fact infinite-dimensional [45]. The conventional one-parametric renormalization subgroup corresponds to a kind of a "constant" shift.

Of this large variety of possibilities, we consider two different regularizations:

c-regularization: boundary condition at the *original* boundary, the shift is "constant", compare with RG of [44] and with [1]. Implies drastic violation of (1.1) in the case of deformed circle. We make this regularization by cutting the integral over

$x \equiv \zeta\bar{\zeta}$ at $1 - c^2$ with non-vanishing c,[3]

$$S_{\mathrm{NG}}\{a,h\} = \int_{|\zeta|^2 \leq 1-c^2} \frac{\sqrt{|\partial H|^2\left(|\partial H|^2 + 4|\partial r|^2\right)}}{r^2}\, d^2\zeta \tag{2.22}$$

$$= \int_{|\zeta|^2 \leq 1-c^2} \frac{\sqrt{|\partial H|^2\left(r^2|\partial H|^2 + |\partial r^2|^2\right)}}{r^3}\, d^2\zeta$$

For

$$r^2 = 1 - |\zeta|^2 + a(\zeta,\bar{\zeta}) \tag{2.23}$$

the action can be expanded as

$$S_{\mathrm{NG}}\{a,h\} = \int_{|\zeta|^2 \leq 1-c^2} \sqrt{\frac{|\partial H|^2\left(|\partial H|^2(1-|\zeta|^2) + \beta|\zeta|^2 + |\partial H|^2 a - \beta\mathcal{D}a + \beta|\partial a|^2\right)}{(1-|\zeta|^2 + a)^{3/2}}}\, d^2\zeta$$

$$= S_{\mathrm{circ}} + S_0\{h\} + S_1\{a,h\} + S_2\{a\} + O(a^{3-j}h^j) \tag{2.24}$$

Here $S_j\{a,h\}$ is of degree j in a and of degree $0,\ldots,2-j$ in h and we specially distinguish the contribution that does not depend on h at all, S_{circ}. As a function of regularization parameter c, each

$$S_j = \frac{1}{c}S_j^{\mathrm{sing}} + S_j^{\mathrm{reg}} + O(c) \tag{2.25}$$

After substitution of (2.19) each S_j becomes a function of the boundary shape $h(z)$:

$$S_j\{a(h),h\} = 2\pi \sum_{k=2}^{\infty} |h_k|^2 \sigma_k^{(j)} + O(h^3) \tag{2.26}$$

---

[3]Following [1], one would also have to introduce a c-dependent factor $\beta(c) = 1 + \beta_1 c + O(c^2)$ into the integrand of action:

$$\frac{\sqrt{|\partial H|^2\left(|\partial H|^2 + 4|\partial r|^2\right)}}{r^2} \to \frac{\sqrt{|\partial H|^2\left(r^2|\partial H|^2 + \beta|\partial r^2|^2\right)}}{r^3} \tag{2.21}$$

This, however, does not lead to any essential effects later on, and we ignore such a modification here. In fact, the role of β_1 would be just to shift $\sigma_j^{\mathrm{reg}} \to \sigma_j^{\mathrm{reg}} + \frac{\beta_1}{2}\sigma_j^{\mathrm{sing}}$ in the formulas below. In fact, β_1 has dimension $length^{-1}$ and can hardly be constant.

Terms of the order $O(c)$ are omitted, with this accuracy one has

$$\sigma_k^{(0)} = k^2 \int_0^{1-c^2} \frac{x^{k-1} dx}{(1-x)^{3/2}} \left(1 - \frac{x}{2} - \frac{x^2}{4}\right) = \frac{k^2}{2c} + I_1,$$

$$\sigma_k^{(1)} = \underbrace{- \frac{kA_k x^{k+1}}{(1-x)^{3/2}}\bigg|_{x=1-c^2}} + \frac{k}{2} \int_0^{1-c^2} \frac{A_k x^{k-1} dx}{(1-x)^{3/2}} \big((k+1)x - 4\big)$$

$$= \underbrace{\frac{k^2(k-3)}{2c} - \frac{k^2(k-1)(k-2)}{3}} + 2I_2,$$

$$\sigma_k^{(2)} = \underbrace{\frac{x^{k-1}A_k^2}{2(1-x)^{5/2}} \big((k-2)x^2 - 2(k-3)x + (k-1)\big)\bigg|_{x=1-c^2}}$$

$$\qquad\qquad\qquad\qquad - \frac{k}{4} \int_0^{1-c^2} \frac{A_k x^{k-1} dx}{(1-x)^{3/2}} \big((k+1)x - 4\big)$$

$$= \underbrace{\frac{3k^2(k-3)^2}{8c} - \frac{k^2(k-1)(k-2)(k-3)}{2}} - I_2 \qquad (2.27)$$

where we "underbraced" the boundary contributions (which come from the integration by parts). The density integrals

$$I_1 = k^2 \int_0^1 \frac{dx}{(1-x)^{3/2}} \left\{ x^{k-1} \left(1 - \frac{x}{2} - \frac{x^2}{4}\right) + \frac{x-2}{4} \right\}$$

$$= \frac{4k^2}{3} \left(1 + 3 \sum_{j=1}^{k-1} (-)^j C_{k-1}^j \frac{2j^2 + 3j - 1}{(2j-1)(2j+1)(2j+3)}\right), \qquad (2.28)$$

$$I_2 = \frac{k}{2} \int_0^1 \frac{dt}{t^2} \Big(\big(1 + (k-1)t\big)(1-t)^{k-1} - (1-t^2)^k\Big) \Big((k-3) - (k+1)t^2\Big) \qquad (2.29)$$

are rather complicated, however, their sum is simple:

$$I_1 + I_2 = -\frac{k(k-1)(k-2)}{2} \qquad (2.30)$$

Non-transcendental boundary terms contribute

$$-\frac{k^2(k-1)(k-2)}{3} - \frac{k^2(k-2)(k-2)(k-3)}{2} = -k(3k-7)\frac{k(k-1)(k-2)}{6} \qquad (2.31)$$

To summarize, the singular term is

$$\sigma_k^{\text{sing}} = \frac{k^2(3k^2 - 14k + 19)}{8c} = \frac{k^2}{2c} + \underbrace{\frac{3k^2(k^2 - 4k + 5)}{8c} - \frac{k^3}{4c}} \qquad (2.32)$$

while the regular term is

$$\sum_{k=2}^{\infty} \sigma_k^{\text{reg}} |h_k|^2 = -\sum_{k=3}^{\infty} (\underbrace{3k^2 - 7k} + 3)\frac{k(k-1)(k-2)}{6} |h_k|^2 \qquad (2.33)$$

– 13 –

μ-regularization: the better one, no direct relation to [44], provides (1.1) for deformed circle up to the coefficient 3 in front of the h^2 terms. The regularization implies just replacing $r^2 \rightarrow r^2 + \mu^2$ in the denominator of the integrand of action:

$$
S_{\mathrm{NG}}\{a, h\} = \int_{|\zeta|^2 \leq 1} \frac{\sqrt{|\partial H|^2 \left(|\partial H|^2 + 4|\partial r|^2\right)}}{r^2 + \mu^2} \, d^2\zeta
$$

$$
= \int_{|\zeta|^2 \leq 1} \frac{\sqrt{|\partial H|^2 \left(r^2|\partial H|^2 + |\partial r^2|^2\right)}}{r(r^2 + \mu^2)} \, d^2\zeta \tag{2.34}
$$

In the case of μ-regularization the boundary (underlined) terms in (2.27) do **not** contribute, and the full answer is

$$
\mathrm{Area}_\mu = \frac{\pi^2}{\mu} \left(1 + \frac{1}{4} \sum_{k=2}^{\infty} k^2 |h_k|^2\right) + \pi \left(I_1 + I_2\right) - 2\pi + O(h^3) \tag{2.35}
$$

The combination $\left(\frac{\pi^2}{\mu} - 2\pi\right)$ here is S_{circ}.

One would expect that the divergent part of the result should be proportional to the length of the wavy circle. This is, indeed, the case for the μ-regularization, since the length of the contour is (see appendix I)

$$
\frac{L}{2\pi} = \oint_\Pi dl = 1 + \sum_{k=2}^{\infty} \frac{k^2|h_k|^2}{4} + O(h^3) \tag{2.36}
$$

and

$$
\mathrm{Area}_\mu = \frac{\pi L}{2\mu} - \pi \sum_{k=3}^{\infty} \frac{k(k-1)(k-2)}{2} |h_k|^2 - 2\pi + O(h^3) \tag{2.37}
$$

Comment. Note that the result for the c-regularization is not same good. It is not proportional to the length and, what is much worse, one can hardly find local boundary counterterms to treat the singularity. Indeed, the only other possible candidate could be integral of logarithm of the scalar curvature

$$
\kappa = \frac{|\mathrm{Im}(\ddot{z}\dot{\bar{z}})|}{|\dot{z}|^3} \tag{2.38}
$$

However, this integral

$$
\oint_\Pi \log \kappa \, dl \sim \sum_{k=2}^{\infty} k^2(k^2 - 4k + 5)|h_k|^2 \tag{2.39}
$$

along with the length term, leave the unbalanced singular term $-\sum \frac{k^3}{4c}|h_k|^2$, see (2.32).

– 14 –

The main lesson one can get from this consideration is that, generally speaking, the result strongly depends on the regularization procedure. However, we expect that for the class of admissible regularizations, i.e. such that the surface terms vanish, the result for the finite part of the minimal area would not depend on the regularization. For example, our c-regularization implied that the boundary condition is set at the original boundary. One can instead shift the boundary conditions to the regularized boundary what effectively corresponds to omitting the surface terms. As we saw above, this would lead to the same result as for the μ-regularization.

2.3 Double countour integral

Above results for the area should now be compared with the (regularized) double loop integral evaluated with the same accuracy up to the h^2-terms. The result for the finite piece is[4]

$$\sum_{k=2}^{\infty} |kh_k|^2 \int \frac{d\varphi}{4\sin^2\varphi} \Big(\cos(2k\varphi) - 2\sigma_k(\varphi) \cos\big((k+1)\varphi\big) + \sigma_k^2(\varphi)\cos(2\varphi) \Big) \qquad (2.42)$$

$$= -2\pi \sum_{k=3}^{\infty} \frac{k(k-1)(k-2)}{6} |h_k|^2$$

Here $\sigma_k(\varphi) \equiv \frac{\sin k\varphi}{k\sin\varphi}$. The divergent piece (see details in (3.8) below) is $\frac{L}{\lambda}$, for example, from

$$\int_0^{2\pi} \frac{B d\varphi}{B \sin^2\varphi + \lambda^2} = \frac{2\pi\sqrt{B}}{\lambda} + O(\lambda) \qquad (2.43)$$

and $\oint \sqrt{B(\Phi)}d\Phi = L$. No term $\quad \lambda^{-1} \oint \log\kappa \, dl \quad$ is present.

2.4 Double integral vs. minimal area

Now, comparing the results of two calculations for the double contour integral and for the area, one can see another problem with the c-regularization: the finite piece it gives has

---

[4]To obtain this result, one uses the following integrals:

$$\int \left(\frac{\sin(k\varphi)}{\sin\varphi}\right)^2 \frac{d\varphi}{2\pi} = k,$$

$$\int \frac{\sin\big((2k+1)\varphi\big) - (2k+1)\sin\varphi}{\sin^3\varphi} \frac{d\varphi}{2\pi} = -2k(k+1) \qquad (2.40)$$

$$\int \frac{\cos\big((k+1)\varphi\big)\sin(k\varphi) - k\sin\varphi}{\sin^3\varphi} \frac{d\varphi}{2\pi} = -k(k+1),$$

$$\int \frac{\cos(2\varphi)\sin^2(k\varphi) - k^2\sin^2\varphi}{\sin^4\varphi} \frac{d\varphi}{2\pi} = -\frac{2k(k^2+2)}{3} \qquad (2.41)$$

nothing to do with the result for the double contour integral (1.6). Indeed,

$$
\begin{aligned}
\sigma_k^{\text{reg}} - D_k &= -(3k^2 - 7k + 4)\frac{k(k-1)(k-2)}{6} + \frac{k(k-1)(k-2)}{6} \\
&= -(3k^2 - 7k + 2)\frac{k(k-1)(k-2)}{6} \\
&= \frac{k(3k-1)(k-1)(k-2)^2}{6}
\end{aligned}
\tag{2.44}
$$

which can not be removed into $\frac{\beta_1}{2}\sigma_k^{\text{sing}}$.

At the same time, the case of Area_μ is much better, though differs by a factor from the double integral:

$$
\text{Area}_\mu = \frac{\pi L}{2\mu} - \pi \sum_{k=3}^{\infty} \frac{k(k-1)(k-2)}{2}|h_k|^2 - 2\pi + O(h^3)
\tag{2.45}
$$

while

$$
\mathrm{D}_\lambda = \frac{2\pi L}{\lambda} - 2(2\pi)^2 \sum_{k=3}^{\infty} \frac{k(k-1)(k-2)}{6}|h_k|^2 - (2\pi)^2 + O(h^3)
\tag{2.46}
$$

By all these reasons, we choose in further calculations only the μ-regularization, keeping in mind that the final result can drastically depend on the regularization, and it is not guaranteed that the μ-regularization is the best/correct one. Anyhow, in this regularization a discrepancy in the overall coefficient occurs in the h^2 terms between (2.45) and (2.46). If our argument at the end of subsection 2.2 about regularization independence of Area_μ is taken seriously, this discrepancy is unavoidable and becomes a kind of anomaly, slightly violating the conjectured form (1.1) of the Alday-Maldacena duality.

2.5 Further corrections

The next step is to check if the same discrepancy is presented in higher orders in h. Naively, one would expect that, in order to obtain h^3-corrections to A_μ, one needs to take into account higher terms in the NG equation etc. However, it turns out that these corrections can be obtained with the already obtained solution (2.11). To see this, let us introduce the notation $S_{(k,l)}$ for the term in action of the order $a^k h^l$. Then, up to the third order, the action is

$$
S = \sum_{l,k=1}^{3} S_{(l,k)}
\tag{2.47}
$$

and the solution to the equation of motion $a^{(1)}$ linear in h is determined from the variation (note that $S_{(1,0)} = 0$)

$$
\left(\frac{\delta S_{(1,1)}}{\delta a} + \frac{\delta S_{(2,0)}}{\delta a} \right) \Bigg|_{a=a^{(1)}} = 0
\tag{2.48}
$$

In order to find the next correction, $a^{(2)}$ one needs to insert $a = a^{(1)} + a^{(2)}$ into the equation

$$\frac{\delta S_{(1,1)}}{\delta a} + \frac{\delta S_{(2,0)}}{\delta a} + \frac{\delta S_{(2,1)}}{\delta a} + \frac{\delta S_{(1,2)}}{\delta a} + \frac{\delta S_{(3,0)}}{\delta a} = 0 \qquad (2.49)$$

etc. Now one needs to calculate the value of action (i.e. the minimal area) on the solution

$$A = \sum_{l,k=1}^{3} S_{(l,k)}\left(a^{(1)} + a^{(2)} + \dots\right) = A^{(2)} + A^{(3)} + \dots \qquad (2.50)$$

Note that the part of the cubic correction $A^{(3)}$ that involves $a^{(2)}$ is linear in it, and, therefore, is proportional to $\left(\frac{\delta S_{(1,1)}}{\delta a} + \frac{\delta S_{(2,0)}}{\delta a}\right)\Big|_{a=a^{(1)}}$ which vanishes by the equation of motion, (2.48). Therefore, only $a^{(1)}$ contributes to the minimal area up to the third order, and one can use the known solution, (2.11) when evaluating the minimal area.

Thus, one just needs to insert solution (2.11) into the action and expand it up to h^3 terms. Similarly, one needs to calculate the double contour integral D_μ up to terms of the same cubic order. This can be done by pen, or with the computer (the corresponding MAPLE programs can be found in appendix IV), the results being formulas (1.7) and (1.6). In the latter case, the h^2 terms and some other contributions of higher order are also presented in order to give a flavour of how they look like. However, in order to include higher order (quartic) terms into the expression for A_μ, one would need to find corrections to the NG solution which is a tedious problem. Here we restrict ourselves only to the cubic terms.

3. Double integral: technicalities

3.1 BDS formula and double loop integral

In the (homogeneous) $n = \infty$ case the BDS formula immediately leads to the double integral, hence, the calculation of [6] can be bypassed. According to [12], the BDS formula is a sum over 4-boxes and each 4-box degenerates into a chordae of the curve Π when $n \to \infty$. The contributions of each 4-box consists of dilogarithmic and logarithmic parts, which degenerate into

$$Li_2\Big(1 - \exp(\tau_l + \tau_s - \tau_{m1} - \tau_{m2})\Big) \overset{n \to \infty}{\longrightarrow} Li_2\left(1 - \exp\left(\delta\phi\delta\phi'\frac{\partial^2 \log t(\phi, \phi')}{\partial\phi\partial\phi'}\right)\right) \qquad (3.1)$$

$$= \delta\phi\delta\phi'\frac{\partial^2 \log t(\phi, \phi')}{\partial\phi\partial\phi'} + O(\delta\phi^3)$$

and

$$(\tau_l - \tau_{m1})(\tau_l - \tau_{m2}) \overset{n \to \infty}{\longrightarrow} \delta\phi\delta\phi'\frac{\partial \log t(\phi, \phi')}{\partial\phi}\frac{\partial \log t(\phi, \phi')}{\partial\phi'} + O(\delta\phi^3) \qquad (3.2)$$

respectively. Here $t = |z(\phi) - z(\phi')|^2$ is the squared length of the chordae, $\tau = \log t$. Adding the dilogarithmic and logarithmic contributions and summing over chordae, one

straightforwardly reproduces the double contour integral

$$\oint \oint_\Pi d\phi d\phi' \left(\frac{\partial^2 \log t(\phi,\phi')}{\partial\phi\partial\phi'} + \frac{\partial \log t(\phi,\phi')}{\partial\phi} \frac{\partial \log t(\phi,\phi')}{\partial\phi'} \right) = \oint \oint_\Pi d\phi d\phi' \frac{1}{t} \frac{\partial^2 t(\phi,\phi')}{\partial\phi\partial\phi'}$$

$$= \oint \oint_\Pi \frac{\mathrm{Re}(d\vec{y}d\vec{y}')}{(\vec{y}-\vec{y}')^2} \qquad (3.3)$$

3.2 Double loop integral for the wavy circle

The necessary ingredients of the double integrand are

$$|z-z'|^2 = 4\sin^2\varphi \bigg\{ 1 + \sum_k k \left(h_k e^{i(k-1)\Phi} + \bar{h}_k e^{-i(k-1)\Phi} \right) \sigma_k(\varphi) \qquad (3.4)$$

$$+ \sum_{k,l} kl h_k \bar{h}_l \sigma_k(\varphi)\sigma_l(\varphi) e^{i(k-l)\Phi} \bigg\}$$

with $\phi = \Phi - \varphi$, $\phi' = \Phi + \varphi$, $\sigma_k(\varphi) = \frac{\sin k\varphi}{k \sin \varphi}$ and

$$\frac{1}{2}(dz d\bar{z}' + d\bar{z} dz') = 2d\Phi d\varphi \bigg\{ \cos(2\varphi) + \sum_k k \left(h_k e^{i(k-1)\Phi} + \bar{h}_k e^{-i(k-1)\Phi} \right) \cos(k+1)\varphi \quad (3.5)$$

$$+ \sum_{k,l} kl h_k \bar{h}_l e^{i(k-l)\Phi} \cos(k+l)\varphi \bigg\}$$

Now one needs to regularize the integral and, then, to calculate it (we remind that $h_1 = \bar{h}_1 = 0$ to simplify formulae)

$$D_\Pi = \frac{1}{2} \oint \oint \frac{dz d\bar{z}' + d\bar{z} dz'}{|z-z'|^2 + \lambda^2} \qquad (3.6)$$

$$= 2 \oint d\Phi \oint d\varphi$$

$$\frac{\cos(2\varphi) + \sum_k k \left(h_k e^{i(k-1)\Phi} + \bar{h}_k e^{-i(k-1)\Phi} \right) \cos(k+1)\varphi + \sum_{k,l} kl h_k \bar{h}_l e^{i(k-l)\Phi} \cos(k+l)\varphi}{4\sin^2\varphi \left(1 + \sum_k k \left(h_k e^{i(k-1)\Phi} + \bar{h}_k e^{-i(k-1)\Phi} \right) \sigma_k(\varphi) + \sum_{k,l} kl h_k \bar{h}_l \sigma_k(\varphi)\sigma_l(\varphi) e^{i(k-l)\Phi} \right) + \lambda^2}$$

$$= 2 \oint d\Phi \oint \frac{d\varphi \cos(2\varphi)}{4\sin^2\varphi + \frac{\lambda^2}{B(\varphi,\Phi)}}$$

$$+ 4\pi \sum_{k=1}^\infty |k h_k|^2 \int \frac{d\varphi}{4\sin^2\varphi} \Big(\cos(2k\varphi) - 2\sigma_k(\varphi)\cos\big((k+1)\varphi\big) + \sigma_k^2(\varphi)\cos(2\varphi) \Big)$$

$$B(\varphi,\Phi) \equiv 1 + \sum_k k \left(h_k e^{i(k-1)\Phi} + \bar{h}_k e^{-i(k-1)\Phi} \right) \sigma_k(\varphi) + \sum_{k,l} kl h_k \bar{h}_l \sigma_k(\varphi)\sigma_l(\varphi) e^{i(k-l)\Phi} \quad (3.7)$$

The second term in (3.7) is finite, we discussed it above in subsection 2.3, while the first

one diverges and is equal to

$$2 \oint d\Phi \oint \frac{d\varphi \cos(2\varphi)}{4\sin^2\varphi + \frac{\lambda^2}{B(\varphi,\Phi)}} = 2 \oint d\Phi \left(\frac{\pi\sqrt{B(0,\Phi)}}{\lambda} - \pi \right) + O(\lambda)$$

$$= 4\pi \left(\frac{\pi}{\lambda}\left[1 + \frac{1}{4}\sum_k k^2 |h_k|^2 \right] - \pi \right) + O(\lambda)$$

$$= \frac{2\pi L}{\lambda} - 4\pi^2 + O(\lambda) \tag{3.8}$$

The constant $4\pi^2$ can be removed, e.g., by the proper choice of β_1 (see footnote 2) and we ignore it from now on.

One can also try other regularizations in calculating the double loop integral. However, as we demonstrate in appendix II, using a counterpart of the c-regularization does not change the result.

4. Minimal area: technicalities

Here we reproduce some technicalities of calculation of the minimal area skipped in section 2.

First of all, we construct the solution to the NG equation in the second order in h and, then, expand the action up to the same second order and reduce the integrals emerging to (2.27).

4.1 Approximate NG equation

We are interested in the contribution $\sim h^2$ to the regularized NG action. Solving the NG equation we obtain

$$a = (1 - \zeta\bar{\zeta})\left(1 + \sum_{k\geq 0} \mathrm{Re}(a_k h_k) + \sum_{k,l\geq 0} \mathrm{Re}(a_{kl}h_k h_l + \tilde{a}_{kl}h_k\bar{h}_l) + O(h^3) \right) \tag{4.1}$$

For $\partial H = 1$:

$$\Delta_{\mathrm{NG}} a \equiv \left(\Delta_0 - \mathcal{D}^2 + \mathcal{D} \right) a = \left(4\partial\bar{\partial} - \zeta^2\partial^2 - 2\zeta\bar{\zeta}\partial\bar{\partial} - \bar{\zeta}^2\bar{\partial}^2 \right) a = O(a^2) \tag{4.2}$$

or, with a^2-terms included,

$$\Delta_{\mathrm{NG}} a + 2\left(\zeta\bar{\partial}a\partial^2 a + \bar{\zeta}\partial a\bar{\partial}^2 a - (\zeta\partial a + \bar{\zeta}\bar{\partial}a)\partial\bar{\partial}a \right) + \frac{a}{1-\zeta\bar{\zeta}}\left(\zeta^2\partial^2 - 2\zeta\bar{\zeta}\partial\bar{\partial} + \bar{\zeta}^2\bar{\partial}^2 \right)a = O(a^3) \tag{4.3}$$

i.e.

$$\Delta_{\mathrm{NG}} a + 2\mathcal{D}(\partial a\bar{\partial}a) - (\mathcal{D}a)\Delta_0 a + a\Delta_0 a - \frac{1}{1-\zeta\bar{\zeta}}a\Delta_{\mathrm{NG}}a = O(a^3) \tag{4.4}$$

(note that $\Delta_0 = \partial_1^2 + \partial_2^2 = 4\partial\bar{\partial}$, $(\partial_i a)^2 = 4\partial a\bar{\partial}a$, $\mathcal{D} = \zeta\partial + \bar{\zeta}\bar{\partial}$ and $\zeta^2\partial^2 + 2\zeta\bar{\zeta}\partial\bar{\partial} + \bar{\zeta}^2\bar{\partial}^2 = \mathcal{D}^2 - \mathcal{D}$).

Now we switch on $\partial H \neq 1$:

$$\frac{|\partial H|^2}{4(1-\zeta\bar\zeta)}\left\{2(|\partial H|^2-1)\Big(\zeta\bar\zeta+2|\partial H|^2(1-\zeta\bar\zeta)\Big)-\zeta\bar\zeta\mathcal{D}\left(\log(|\partial H|^2)\right)+\right. \tag{4.5}$$

$$+\Delta_{\mathrm{NG}}a+(|\partial H|^2-1)(1-\zeta\bar\zeta)\Delta_0 a+$$

$$+\left(\mathcal{D}a+\frac{\zeta\bar\zeta a}{1-\zeta\bar\zeta}\right)\Big(2(1-|\partial H|^2)+\mathcal{D}\left(\log(|\partial H|^2)\right)\Big)$$

$$+\zeta\bar\zeta\Big(\partial a\bar\partial\log|\partial H|^2+\bar\partial a\partial\log|\partial H|^2\Big)+$$

$$\left.+2\mathcal{D}(\partial a\bar\partial a)-(\mathcal{D}a)\Delta_0 a+a\Delta_0 a-\frac{1}{1-\zeta\bar\zeta}a\Delta_{\mathrm{NG}}a\right\}=O(a^k h^{3-k})$$

The a-independent piece in curved brackets in (4.5) is

$$8\sum_{k=1}^\infty \mathrm{Re}\left(kh_k\zeta^{k-1}\right)\left(1-\frac{k+1}{4}\zeta\bar\zeta\right)+O(h^2)=2\Delta_{\mathrm{NG}}\left(\sum_{k=1}^\infty\mathrm{Re}\left(\bar\zeta h_k\zeta^k\right)\right) \tag{4.6}$$

Thus (see (2.19))

$$a(\zeta,\bar\zeta)=2\sum_{k=1}^\infty\mathrm{Re}\left(h_k\zeta^{k-1}\right)\left(\frac{F_{k-1}(\zeta\bar\zeta)}{F_{k-1}(1)}-\zeta\bar\zeta\right)+O(h^2) \tag{4.7}$$

(this quantity vanishes when $\zeta\bar\zeta=1$ and $a(\zeta,\bar\zeta)+\bar\zeta h(\zeta)+\zeta\bar h(\bar\zeta)=a(\zeta,\bar\zeta)+2\sum_{k=1}^\infty\mathrm{Re}(\bar\zeta h_k\zeta^k)$ is a zero-mode of Δ_{NG}).

4.2 NG action on NG solution up to the h^2 terms

$$\int\frac{\sqrt{|\partial H|^2(|\partial H|^2+4\partial r\bar\partial r)}}{r^2}d^2\zeta=\int\frac{\sqrt{|\partial H|^2\Big(|\partial H|^2(1-\zeta\bar\zeta+a)+\big|\zeta-\bar\partial a\big|^2\Big)}}{r^3}d^2\zeta \tag{4.8}$$

$$=\int\frac{\sqrt{|\partial H|^2\Big(|\partial H|^2+\zeta\bar\zeta(1-|\partial H|^2)+a|\partial H|^2-\mathcal{D}a+|\partial a|^2\Big)}}{(1-\zeta\bar\zeta)^{3/2}}$$

$$\times\left(1-\frac{3a}{2(1-\zeta\bar\zeta)}+\frac{15a^2}{8(1-\zeta\bar\zeta)^2}+O(a^3)\right)d^2\zeta$$

At the moment we ignore regularization, it can be easily restored. Under the root sign one has up to the second order in h:

$$\Big(1+(\partial h+\overline{\partial h})+|\partial h|^2\Big)\times$$

$$\times\Big(1+(\partial h+\overline{\partial h})(1-\zeta\bar\zeta)+|\partial h|^2(1-\zeta\bar\zeta)+(a-\mathcal{D}a)+|\partial a|^2+a(\partial h+\overline{\partial h})\Big) \tag{4.9}$$

$$=1+\Big(a-\mathcal{D}a+(\partial h+\overline{\partial h})(2-\zeta\bar\zeta)\Big)$$

$$+\Big((\partial h+\overline{\partial h})^2(1-\zeta\bar\zeta)+|\partial h|^2(2-\zeta\bar\zeta)+2(\partial h+\overline{\partial h})a-(\partial h+\overline{\partial h})\mathcal{D}a+|\partial a|^2\Big)$$

and the square root is equal to

$$1 + \frac{1}{2}\Big(a - \mathcal{D}a + (\partial h + \overline{\partial h})(2 - \zeta\bar\zeta)\Big) + \tag{4.10}$$

$$+\frac{1}{8}\Big(4(\partial h + \overline{\partial h})^2(1 - \zeta\bar\zeta) + 4|\partial h|^2(2 - \zeta\bar\zeta) + 8(\partial h + \overline{\partial h})a - 4(\partial h + \overline{\partial h})\mathcal{D}a +$$

$$+4|\partial a|^2 - \big(a - \mathcal{D}a + (\partial h + \overline{\partial h})(2 - \zeta\bar\zeta)\big)^2\Big)$$

$$= 1 + \frac{1}{2}\Big(a - \mathcal{D}a + (\partial h + \overline{\partial h})(2 - \zeta\bar\zeta)\Big) + \frac{1}{8}\Big(4|\partial a|^2 - (a - \mathcal{D}a)^2\Big) +$$

$$+\frac{1}{8}\Big(4|\partial h|^2(2 - \zeta\bar\zeta) - (\zeta\bar\zeta)^2(\partial h + \overline{\partial h})^2 + 2(2 + \zeta\bar\zeta)(\partial h + \overline{\partial h})a - 2\zeta\bar\zeta(\partial h + \overline{\partial h})\mathcal{D}a\Big)$$

Now we substitute

$$\partial h = \sum_{k=1}^{\infty} k h_k \zeta^{k-1},$$

$$\partial h + \overline{\partial h} = \sum_{k=1}^{\infty} 2k \mathrm{Re}(h_k \zeta^{k-1}),$$

$$a = 2 \sum_{k=1}^{\infty} \mathrm{Re}(h_k \zeta^{k-1}) A_k(\zeta\bar\zeta),$$

$$\mathcal{D}a = 2 \sum_{k=1}^{\infty} \mathrm{Re}(h_k \zeta^{k-1})\Big((k-1)A_k(\zeta\bar\zeta) + 2\zeta\bar\zeta A_k'(\zeta\bar\zeta)\Big),$$

$$\mathcal{D}a - a = 2 \sum_{k=1}^{\infty} \mathrm{Re}(h_k \zeta^{k-1})\Big((k-2)A_k(\zeta\bar\zeta) + 2\zeta\bar\zeta A_k'(\zeta\bar\zeta)\Big),$$

$$\partial a = \sum_{k=2}^{\infty} (k-1)h_k \zeta^{k-2} A_k(\zeta\bar\zeta) + 2\zeta\bar\zeta \sum_{k=1}^{\infty} \mathrm{Re}(h_k \zeta^{k-2}) A_k'(\zeta\bar\zeta) \tag{4.11}$$

(note that there is no singular term with ζ^{-1} in the last line) and perform angular integration. Then the term linear in h vanishes (it is proportional to h_1 and $\bar h_1$ which we put equal to zero), and the h^2-term in the action is proportional to:

$$\sum_{k=2}^{\infty} |h_k|^2 \int_0^1 \frac{\rho d\rho}{(1-\rho^2)^{3/2}} \left\{ \rho^{2(k-2)}\left(\frac{(k-1)^2}{2}A_k^2 + (k-1)\rho^2 A_k A_k' + \rho^4 (A_k')^2\right) \right. \tag{4.12}$$

$$-\frac{1}{4}\rho^{2(k-1)}\Big((k-2)A_k + 2\rho^2 A_k'\Big)^2 +$$

$$+k^2\rho^{2(k-1)}\Big(1 - \frac{1}{2}\rho^2\Big) - \frac{1}{4}k^2\rho^4\rho^{2(k-1)}$$

$$+\Big(1 + \frac{1}{2}\rho^2\Big)k\rho^{2(k-1)}A_k - \frac{1}{2}k\rho^{2k}\Big((k-1)A_k + 2\rho^2 A_k'\Big) -$$

$$+3\frac{1}{1-\rho^2}\Big(\frac{1}{2}\rho^{2(k-1)}A_k\big((k-2)A_k + 2\rho^2 A_k'\big) -$$

$$\left. -\Big(1 - \frac{1}{2}\rho^2\Big)\rho^{2(k-1)}k A_k\Big) + \frac{15}{4(1-\rho^2)^2}\rho^{2(k-1)}A_k^2 \right\}$$

The terms independent on A_k and their derivatives are collected into $\sigma_k^{(0)}$, those linear in A_k and their derivatives into $\sigma_k^{(1)}$, and the quadratic terms into $\sigma_k^{(2)}$

$$(4.12) = \sigma_k^{(0)} + \sigma_k^{(1)} + \sigma_k^{(2)} \tag{4.13}$$

4.3 Calculating integrals

Now we calculate the integral (4.12). To this end, note that since F_k's satisfy the equations

$$x(1-x)F_k''(x) + \left(k+1 - \left(k+\frac{1}{2}\right)x\right)F_k'(x) - \frac{k(k-1)}{4}F_k(x) = 0 \tag{4.14}$$

the functions $A_k(x) = \frac{F_{k-1}(x)}{F_{k-1}(1)} - x$ satisfy

$$x(1-x)A_k'' + \left(k - \left(k-\frac{1}{2}\right)x\right)A_k' - \frac{(k-1)(k-2)}{4}A_k = \frac{k(k+1)}{4}x - k \tag{4.15}$$

The terms

$$\sigma_k^{(0)} = \int \frac{dx}{(1-x)^{3/2}}k^2 x^{k-1}\left(1 - \frac{x}{2}x - \frac{x^2}{4}\right) \tag{4.16}$$

and

$$\sigma_k^{(1)} = \int \frac{dx}{(1-x)^{3/2}}\left\{-kx^{k+1}A_k' + x^{k-1}\left(k\left(1+\frac{1}{2}x\right) - \frac{1}{2}k(k-1)x - \frac{3(2-x)k}{2(1-x)}\right)A_k\right\} \tag{4.17}$$

are immediately[5] reduced to (2.27), while in order to calculate

$$\sigma_k^{(2)} = \int \frac{dx}{(1-x)^{3/2}}\left\{x^k(1-x)(A_k')^2 + 2x^{k-1}\left(\frac{k-1}{2} - \frac{(k-2)x}{2} + \frac{3x}{2(1-x)}\right)A_k A_k' + x^{k-2}\left(\frac{(k-1)^2}{2} - \frac{(k-2)^2x}{4} + \frac{3(k-2)x}{2(1-x)} + \frac{15x}{4(1-x)^2}\right)A_k^2 - \tag{4.19}\right.$$

we integrate the first term by parts and make use of (4.15):

$$\int \frac{dx}{(1-x)^{3/2}}x^k(1-x)(A_k')^2 = \lim_{x\to 1-0}\frac{x^k A_k A_k'}{\sqrt{1-x}} \tag{4.20}$$
$$- \int \frac{x^{k-1}dx}{(1-x)^{3/2}}\left(x(1-x)A_k'' + \left((1-x)k + \frac{x}{2}\right)A_k'\right)A_k$$
$$= \lim_{x\to 1-0}\frac{x^k A_k A_k'}{\sqrt{1-x}}$$
$$- \frac{1}{4}\int \frac{x^{k-1}dx}{(1-x)^{3/2}}\left((k-1)(k-2)A_k^2 + \left(k(k+1)x - 4k\right)A_k\right)$$

[5]Since

$$-\int \frac{dx}{(1-x)^{3/2}}kx^{k+1}A_k' = -\lim_{x\to 1-0}\frac{kA_k x^{k+1}}{(1-x)^{3/2}} + \int \frac{A_k x^k dx}{(1-x)^{3/2}}\left(k(k+1) + \frac{3kx}{2(1-x)}\right), \tag{4.18}$$

– 22 –

Integrating by parts the second term in (4.19) we obtain:

$$
\int \frac{x^{k-1}dx}{(1-x)^{3/2}} \left(\frac{k-1}{2} - \frac{(k-2)x}{2} + \frac{3x}{2(1-x)} \right) (2A_k A_k')
$$
$$
= \lim_{x \to 1-0} \frac{x^{k-1}A_k^2}{(1-x)^{3/2}} \left(\frac{k-1}{2} - \frac{(k-2)x}{2} + \frac{3x}{2(1-x)} \right) -
$$
$$
- \int \frac{x^{k-2}dx}{(1-x)^{3/2}} \left(\frac{(k-1)^2}{2} - \frac{k(k-2)x}{2} + \frac{3kx}{2(1-x)} \right.
$$
$$
\left. + \frac{3x}{2(1-x)} \left(\frac{k-1}{2} - \frac{(k-2)x}{2} \right) + \frac{15x^2}{4(1-x)^2} \right) A_k^2 \qquad (4.21)
$$

Collecting all the terms with A_k^2,

$$
\int \frac{A_k^2 x^{k-2}dx}{(1-x)^{3/2}} \left\{ \left(\frac{(k-1)^2}{2} - \frac{(k-2)^2 x}{4} + \frac{3(k-2)x}{2(1-x)} + \frac{15x}{4(1-x)^2} \right) - \frac{(k-1)(k-2)x}{4} - \right. \qquad (4.22)
$$
$$
- \left(\frac{(k-1)^2}{2} - \frac{k(k-2)x}{2} + \frac{3kx}{2(1-x)} \right.
$$
$$
\left. \left. + \frac{3x}{2(1-x)} \left(\frac{k-1}{2} - \frac{(k-2)x}{2} \right) + \frac{15x^2}{4(1-x)^2} \right) \right\}
$$
$$
= \int \frac{A_k^2 x^{k-1}dx}{(1-x)^{3/2}} \left\{ \frac{k-2}{4} \left(-(k-2) - (k-1) + 2k \right) \right.
$$
$$
\left. + \frac{3}{2(1-x)} \left(k-2 + \frac{5}{2} - k - \left(\frac{k-1}{2} - \frac{(k-2)x}{2} \right) \right) \right\}
$$
$$
\int \frac{A_k^2 x^{k-1}dx}{(1-x)^{3/2}} \left\{ \frac{3(k-2)}{4} + \frac{3}{4(1-x)} \left(1 - (k-1) + (k-2)x \right) \right\} = 0
$$

and one remains only with

$$
\lim_{x \to 1-0} \frac{x^k A_k A_k'}{\sqrt{1-x}} - \frac{1}{4} \int \frac{x^{k-1}dx}{(1-x)^{3/2}} \left(k(k+1)x - 4k \right) A_k \qquad (4.23)
$$

which is the same as $\sigma_k^{(2)}$ in (2.27), since the boundary term vanishes.

5. Conformal symmetry

In our ADS_3-restricted problem the global conformal symmetry of [5, 16, 28, 30, 31] reduces to SL(2) with three complex-valued generators. In what follows we use the formulation of [31].

5.1 SL(2) action at the boundary

When acting on a functional $F\{z(s)\}$ of parameterized curve $\Pi : S^1 \to C$, the three

generators are

$$\hat{J}_-F = \oint \frac{\delta F}{\delta z(s)} ds,$$

$$\hat{J}_0 F = \oint z \frac{\delta F}{\delta z(s)} ds,$$

$$\hat{J}_+F = \oint z^2 \frac{\delta F}{\delta z(s)} ds \tag{5.1}$$

There are additional three complex-conjugate operators. Since in [31] the general situation (beyond complex plane) is considered, the third generator in (5.1) was written in a more general form

$$\vec{\hat{\mathcal{J}}}_- = \oint ds \frac{\delta}{\delta \vec{y}(s)},$$

$$\hat{\mathcal{J}}_0 = \oint ds \left(\vec{y}(s) \frac{\delta}{\delta \vec{y}(s)} \right),$$

$$\vec{\hat{\mathcal{J}}}_+ = \oint ds \left\{ 2 \vec{y}(s) \left(\vec{y}(s) \frac{\delta}{\delta \vec{y}(s)} \right) - \vec{y}^2(s) \frac{\delta}{\delta \vec{y}(s)} \right\} \tag{5.2}$$

They are $\vec{\mathcal{J}}_- = (J_-, \bar{J}_-)$, $\mathcal{J}_0 = J_0 + \bar{J}_0$ and $\vec{\mathcal{J}}_+ = (J_+, \bar{J}_+)$ in our situation.

We now need to express these generators in terms of h_k variables. From $z = \zeta + \sum_k h_k \zeta^k$, $\bar{z} = \bar{\zeta} + \sum_k \bar{h}_k \bar{\zeta}^k$ and

$$\delta F = \oint \frac{\delta F}{\delta z(s)} \delta z(s) ds + \oint \frac{\delta F}{\delta \bar{z}(s)} \delta \bar{z}(s) ds$$

$$= \sum_k \delta h_k \oint \frac{\delta F}{\delta z(s)} \zeta^k(s) ds + \sum_k \delta \bar{h}_k \oint \frac{\delta F}{\delta \bar{z}(s)} \bar{\zeta}^k(s) ds \tag{5.3}$$

we conclude that

$$\oint \frac{\delta F}{\delta z(s)} \zeta^k(s) ds = \frac{\partial F}{\partial h_k}, \quad \oint \frac{\delta F}{\delta \bar{z}(s)} \bar{\zeta}^k(s) ds = \frac{\partial F}{\partial \bar{h}_k} \tag{5.4}$$

Therefore

$$\hat{J}_- = \frac{\partial}{\partial h_0},$$

$$\hat{J}_0 = \frac{\partial}{\partial h_1} + \sum_{k=0}^{\infty} h_k \frac{\partial}{\partial h_k},$$

$$\hat{J}_+ = \frac{\partial}{\partial h_2} + 2 \sum_{k=0}^{\infty} h_k \frac{\partial}{\partial h_{k+1}} + \sum_{k,l=0}^{\infty} h_k h_l \frac{\partial}{\partial h_{k+l}} \tag{5.5}$$

5.2 Invariance properties of h-series: a surprise

It is easy to check that (1.6) is invariant under these SL(2) transformations, while (1.7) is not. Indeed, $\hat{J}_-$ annihilates all h-series that do not contain h_0 — and both (1.6) and (1.7) belong to this class.

The relevant properties of the coefficients in (1.6) are:

$$\begin{aligned}
C_{1k} &= B_k, & C_{11} = C_{12} &= 0, & C_{2k} &= 2B_{k+1}, \\
V_{1kl} &= 2C_{kl}, & V_{2kl} + 2A_{k+l} &= 2\left(C_{k,l+1} + C_{k+1,l}\right) & & \\
U_{ij;1l} &= C_{ij}, & U_{ij;2l} &= 2C_{ij} & &
\end{aligned} \tag{5.6}$$

They are indeed satisfied by the coefficients $B^{(D)}$, $C^{(D)}$, $V^{(D)}$ and $U^{(D)}$ in (1.6). At the same time for (1.7) $C_{22}^A \neq 2B_3^A$!

This fact is somewhat surprising because one could expect the opposite result: the double integral D_Π is not *a priori* annihilated by $\hat{J}_{+1}$, while A_Π is shown to be invariant [31]. In particular, the BDS formula is known to satisfy (anomalous) conformal Ward identities for all Π [5, 16, 28, 30, 31]. Indeed, dilogarithms in the BDS formula [36] depend only on invariant cross-ratios, while logarithms reproduce the anomaly part of the Ward identity.

5.3 Invariance of the double integral

To explain the invariance of the double integral, one should note that the integrand in D_Π is obviously not invariant under the projective transformations generated by (5.1), instead it changes by a total derivative. Therefore, as soon as the integral diverges, one has to be careful with its invariance. Indeed, one can easily see the divergent part is not projective-invariant: it is proportional to the curves length $L = \oint dl = \oint \sqrt{\dot{z}\dot{\bar{z}}}ds$, which transforms as follows:

$$\hat{J}_- L = 0, \qquad \hat{J}_0 L = \frac{L}{2}, \qquad \hat{J}_+ L = \oint z\,dl \tag{5.7}$$

as can be read off from formulae (5.1).

At the same time, this quite formal calculation can be confirmed from explicit manipulations with the h-series. When J_0 from (5.5) acts on L which is given by formula (1.16) with h_1 and $\bar{h}_1$ switched on, then it converts the typical term in the h series for L,

$$\sqrt{(1 + h_1)(1 + \bar{h}_1)}\left(\frac{h}{1 + h_1}\right)^p\left(\frac{\bar{h}}{1 + \bar{h}_1}\right)^q \tag{5.8}$$

into

$$\hat{J}_0\sqrt{(1 + h_1)(1 + \bar{h}_1)}\left(\frac{h}{1 + h_1}\right)^p\left(\frac{\bar{h}}{1 + \bar{h}_1}\right)^q \tag{5.9}$$
$$= \left[p - \left(p - \frac{1}{2}\right)\right]\sqrt{(1 + h_1)(1 + \bar{h}_1)}\left(\frac{h}{1 + h_1}\right)^p\left(\frac{\bar{h}}{1 + \bar{h}_1}\right)^q$$

i.e.

$$\hat{J}_0 L = \frac{L}{2} \tag{5.10}$$

as required in (5.7).

Similarly, one can use the explicit form of $\hat{J}_+$ in terms of h,

$$\hat{J}_+ = h_0^2\frac{\partial}{\partial h_0} + 2h_0\hat{J}_0 + (1 + h_1)^2\frac{\partial}{\partial h_2} + 2(1 + h_1)\sum_{k=2} h_k\frac{\partial}{\partial h_{k+1}} + \sum_{k,l\geq 2} h_k h_l\frac{\partial}{h_{k+l}} \tag{5.11}$$

and act with it on L from (1.16),

$$\frac{1}{2\pi}\oint dl = |1+h_1| + \frac{1}{4}\sum_{k=2}\frac{k^2|h_k|^2}{|1+h_1|} \tag{5.12}$$

$$-\frac{1}{16}\sum_{k,l=2}^{\infty} kl(k+l-1)\left[\frac{h_k h_l \bar{h}_{k+l-1}}{(1+h_1)\,|1+h_1|} + \frac{\bar{h}_k \bar{h}_l h_{k+l-1}}{(1+\bar{h}_1)\,|1+h_1|}\right] + \dots$$

to obtain

$$\hat{J}_+\frac{L}{2\pi} = 2h_0\left(\hat{J}_0\frac{L}{2\pi}\right) + (1+h_1)^2\left[\frac{2^2}{4}\bar{h}_2 - \frac{1}{16}\frac{2\cdot2}{(1+h_1)|1+h_1|}\sum_{k=2} k(k+1)h_k\bar{h}_{k+1}\right] +$$

$$+2(1+h_1)\frac{1}{4|1+h_1|}\sum_{k=2}(k+1)^2 h_k\bar{h}_{k+1}$$

$$= \frac{h_0}{2\pi}\oint dl + (1+h_1)^2\frac{\bar{h}_2}{|1+h_1|} + \sum_{k=2}\frac{(k+1)(k+2)}{4}\frac{1+h_1}{|1+h_1|}h_k\bar{h}_{k+1}$$

$$= \frac{1}{2\pi}\oint z\,dl \tag{5.13}$$

in accordance with (5.7). Note that this calculation depends on the explicit form of h^3-terms.

5.4 On symmetries of the minimal area

First of all, the r.h.s. of the anomalous Ward identity (A.19) in [31] vanishes in our smooth $n=\infty$ limit. Therefore, according to [31] the minimal area is conformal invariant! – what seems to contradict apparent non-invariance of A_Π.

For an *a priori* check of the symmetry of the minimal action one needs to extend the action of SL(2) from the boundary to entire AdS space. The group action is [31]:

$$r \to \frac{r}{1+2\vec{\beta}\vec{y}+\vec{\beta}^2(r^2+\vec{y}^2)},$$

$$\vec{y} \to \frac{\vec{y}+\vec{\beta}(r^2+\vec{y}^2)}{1+2\vec{\beta}\vec{y}+\vec{\beta}^2(r^2+\vec{y}^2)} \tag{5.14}$$

At the boundary $r^2=0$ it reduces to the projective transformation $z \to z+\bar{\beta}z^2+O(\beta)$. The problem is that for $r^2 \neq 0$ the action of $\hat{J}_+$ on z transforms it into non-holomorphic function of ζ. Application of the Gauss-Riemann decomposition is needed to restore holomorphicity, what can imply a more sophisticated action on h-variables beyond the boundary. It can happen that such modifications involve μ-linear terms, which can generate μ-finite corrections from the variation of L/μ contributions. This is also a kind of anomaly — which needs to be studied more accurately. This anomaly in conformal symmetry (5.5) is a part of a larger anomaly for $n=\infty$ discovered in this paper, which, in its turn, generalizes the Alday-Maldacena result, [23]. A similar anomaly for $n=6$ was recently found in [30], see also a fresh additional evidence in [40, 41].

Acknowledgments

A. h-series representation of reparametrization invariants at the boundary

Here we express the circumference of the deformed circle and the integral of logarithm of its curvature as functions of coefficients h_k up to the second order in these coefficients (we still put $h_1 = \bar{h}_1 = 0$).

With the conformal map

$$z = e^{i\varphi} + \sum_{k=0}^{\infty} h_k e^{ik\varphi} \tag{A.1}$$

the square of length element is

$$dl^2 = \left| \frac{dz}{d\varphi} \right|^2 = \left| 1 + \sum k h_k e^{i(k-1)\varphi} \right|^2$$
$$= 1 + 2 \sum_{k=1}^{\infty} \mathrm{Re} \left(k h_k e^{i(k-1)\varphi} \right) + \sum_{k,l=1}^{\infty} kl \mathrm{Re} h_k \bar{h}_l e^{i(k-l)\varphi} \tag{A.2}$$

Integration along the circle over $\frac{d\varphi}{2\pi}$ converts the sums of exponentials in the following way:

$$\sum_{k=2}^{\infty} f(k) \mathrm{Re} \left(h_k e^{i(k-1)\varphi} \right) \longrightarrow 0 \tag{A.3}$$

$$\sum_{k,l=1}^{\infty} \mathrm{Re} f(k,l) h_k \bar{h}_l e^{i(k-l)\varphi} \longrightarrow \sum_{k=1}^{\infty} f(k,k) |h_k|^2$$

$$\left\{ \sum_{k=1}^{\infty} f(k) \mathrm{Re} \left(h_k e^{i(k-1)\phi} \right) \right\} \left\{ \sum_{k=1}^{\infty} g(k) \mathrm{Re} \left(h_k e^{i(k-1)\phi} \right) \right\} \longrightarrow \frac{1}{2} \sum_{k=1}^{\infty} f(k) g(k) |h_k|^2$$

Keeping this in mind, one gets

$$\frac{1}{2\pi} L = \int dl = 1 + \frac{1}{4} \sum_{k=1}^{\infty} k^2 |h_k|^2 + O(h^3) \tag{A.4}$$

Proceed now to curvature and its derivatives. The first local reparametrization invariant (scalar) of a curve is its scalar curvature,

$$\kappa = \frac{\mathrm{Im}\left(\ddot{z}\dot{\bar{z}}\right)}{|\dot{z}\dot{\bar{z}}|^{3/2}} \tag{A.5}$$

and

$$\frac{1}{2\pi}\int \log\kappa \, dl = -\frac{1}{4}\sum_{k=1}^{\infty}(k^2 - 4k + 5)k^2|h_k|^2 \tag{A.6}$$

Similarly, one can calculate $d\kappa/d\varphi$, $\dot{\kappa} \equiv d\kappa/dl$ and the integral of square of this latter, the result reads

$$\frac{1}{2\pi}\int \left(\frac{d\kappa}{dl}\right)^2 dl = \frac{1}{2}\sum_{k=3}^{\infty}\Big(k(k-1)(k-2)\Big)^2|h_k|^2 \tag{A.7}$$

B. An alternative calculation of D_Π by r'/r regularization

In this appendix, we compute the double contour integral D_Π to the quadratic order, regularizing the integral by making the relative size of the two radii r, r' associated with the two circular line integrals in ζ plane different from unity. This is a version of c-regularization, an alternative regularization to the "λ" regularization in the text. Let $rr' = 1$, $z = H(\zeta) = \zeta + h(\zeta)$.

$$D_\Pi = \oint_{\Pi_r}\oint_{\Pi_{r'}}\frac{\frac{1}{2}(dz d\bar{z}' + d\bar{z} dz')}{(z-z')(\bar{z}-\bar{z}')} = D_\Pi^{(0)} + D_\Pi^{(1)} + D_\Pi^{(2)} + O(h^3), \tag{B.1}$$

where $D_\Pi^{(i)}, i = 0, 1, 2$ denote order h^0, h^1 and h^2 contribution to D_Π respectively. It is immediate to see that $D_\Pi^{(1)} = 0$ and

$$D_\Pi^{(0)} = \oint_{|\zeta|=r}\oint_{|\zeta'|=r'}\frac{\frac{1}{2}(d\zeta d\bar{\zeta}' + d\bar{\zeta} d\zeta')}{(\zeta-\zeta')(\bar{\zeta}-\bar{\zeta}')}. \tag{B.2}$$

After splitting the double integral into that over total and relative angles, Φ and φ, (B.2) becomes a simple Poisson integral:

$$2(2\pi)a\int_{-\pi}^{\pi}\frac{\cos\varphi d\varphi}{1-2a\cos\varphi+a^2} = 2(2\pi)^2\left(\frac{1}{1-a^2}-1\right). \tag{B.3}$$

where $a \equiv \frac{r'}{r}$. As for $D_\Pi^{(2)}$, after some calculation, we obtain

$$D_\Pi^{(2)} = \sum_{k,\ell}h_k\bar{h}_\ell\Bigg[\frac{1}{2}\oint\oint d\zeta d\bar{\zeta}'\frac{\zeta^{k-1}\bar{\zeta}'^{\ell-1}f_k(\zeta'/\zeta)f_\ell(\bar{\zeta}/\bar{\zeta}')}{(\zeta-\zeta')(\bar{\zeta}-\bar{\zeta}')} \tag{B.4}$$

$$+\frac{1}{2}\oint\oint d\bar{\zeta}d\zeta'\frac{\zeta'^{k-1}\bar{\zeta}^{\ell-1}f_k(\zeta/\zeta')f_\ell(\bar{\zeta}'/\bar{\zeta})}{(\zeta-\zeta')(\bar{\zeta}-\bar{\zeta}')}\Bigg],$$

where

$$f_k(x) = \frac{1 - x^k}{1 - x} - k. \tag{B.5}$$

We put $a = 1$, since the integral is finite at this point. Making a change of variables $\zeta = e^{i(\Phi - \frac{1}{2}\varphi)}, \zeta' = e^{i(\Phi + \frac{1}{2}\varphi)}, w = e^{i\varphi}$, and carrying out the $d\Phi$ integral, we obtain

$$-(2\pi)^2 \sum_k |h_k|^2 \left[\oint \frac{dw}{2\pi i} \frac{w^{-k} f_k(w)^2}{(1 - w)^2} + c.c. \right] = -2(2\pi)^2 \sum_k |h_k|^2 \left(\sum_{i=0}^{k-1} c_i c_{k-1-i} \right)$$

$$= -2(2\pi)^2 Q_\Pi^{(2)}, \tag{B.6}$$

where c_ℓ with $c_i = -(k-1) + i$, for $0 \le i \le k - 1$ are the Taylor coefficients

$$\frac{f_k(x)}{(1 - x)} = \sum_{n=0}^{\infty} c_n x^n. \tag{B.7}$$

Eq. (B.6) agrees with the result (1.6) of calculations in the λ regularization.

To summarize,

$$D_\Pi = 2(2\pi)^2 \left(\frac{1}{1 - a^2} - 1 \right) - 2(2\pi)^2 Q_\Pi^{(2)} + O(h^3). \tag{B.8}$$

Higher order computation can be carried out as is in the main text.

C. Circle vs. rectangular

In this appendix, we comment on technical differences between the long rectangular that was considered in [23] and the deformed circle we consider in the paper.

C.1 Asymptotic behavior of r near the boundary

First of all, let us consider the behaviour of solution to the NG equation. From (2.2) in the leading order in $y_\perp$ and $y_{||}$ we get

$$r = \sqrt{\frac{2y_\perp - \kappa y_{||}^2}{\kappa}} \tag{C.1}$$

where κ is the curvature (inverse radius of the tangent circle) at the given point of the boundary. This can be considered as a limit near the boundary of exact circle solution (2.4),

$$r = \sqrt{\kappa^{-2} - (\kappa^{-1} - y_\perp)^2 - y_{||}^2} \tag{C.2}$$

Technically the contribution to (2.2) in this order comes from

$$\partial_\perp r = \frac{1}{\kappa r},$$
$$\partial_\perp^2 r = -\frac{1}{\kappa^2 r^3},$$
$$\partial_{||}^2 r = -\frac{1}{r} + O(y_{||}^2) \tag{C.3}$$

the first derivative $\partial_{||}r$ and the mixed derivative $\partial_\perp\partial_{||}r$ are proportional to $y_{||}$ and can be neglected. Then the relevant terms in (2.2) are

$$2(\partial_\perp r)^2 + r\partial_\perp^2 r + r(\partial_\perp r)^2\partial^2 r = r(\partial_\perp r)^2\partial_\perp^2 r \qquad (C.4)$$

Since at $r \to 0$ the second derivative $\partial_{||}^2 r \ll \partial_\perp^2 r$, it can be neglected in the $r\partial^2 r$ term, but it contributes to the r^4 terms, because it is multiplied by a large factor $\partial_\perp r$. These two r^4 terms actually combine into $r(\partial_\perp r)^2\partial_{||}^2 r$ at the l.h.s. and this contribution is crucial for (C.2) to be a solution to (2.2): the three terms at the l.h.s. contribute $2 - 1 - 1 = 0$.

Already from this calculus it is clear that things will go wrong if (C.2) does not depend on $y_{||}$. This happens when the boundary straightens, $\kappa = 0$, even at a single point - nothing to say about the boundary containing entire straight segments like in [23].[6] The problem is already seen in (C.2): κ enters also as a normalization factor and stands in the denominator. Clearly, at $\kappa = 0$ asymptotics (C.2) is seriously modified, actually it is substituted by

$$r \sim \sqrt[3]{y_\perp}, \qquad (C.5)$$

(note that $\sqrt[3]{y_\perp} \gg \sqrt{y_\perp}$ at small $y_\perp$). The interpolating formula

$$2y_\perp - \kappa y_{||}^2 = \kappa r^2 + \text{const} \cdot r^3 + O(r^4) \qquad (C.6)$$

The situation gets even more tricky if convexity of the curve $\bar{\Pi}$ is changed: solution (C.2) turns imaginary at the other side of the boundary — i.e. simply fails to exist. This means that, near the boundary, the minimal surface is locally bent towards the center of curvature of the boundary.

In any case we see that at $n = \infty$ the $\bar{\Pi}$ with some straight segments is a kind of a very special limit, considerably different from generic situation. This can imply that the long-rectangular example of [23], despite its seeming simplicity can actually be non-trivial and require a more serious analysis. We, however, restrict ourselves to a brief reminder of that example in the next subsection.

C.2 An example of rectangular

We calculate here the minimal area of the rectangular and demonstrate it does not look like the double contour integral [46].

We consider a very long rectangular of the length $L_{||}$ and the width L so that the solution to the NG equations depends on the only perpendicular variable $y_\perp = y$. Then, the solution $r(y)$ is easier written in terms of the inverse function

$$y(r) = \int_0^r \frac{\xi^2 d\xi}{\sqrt{C^4 - \xi^4}} = -CD\left(\arcsin\frac{r}{C}, i\right)$$

---

[6]It deserves emphasizing that we speak here about a straight segment in *projection* $\bar{\Pi}$ in the $n = \infty$ limit: this argument is non-applicable neither to the light-like straight segments which compose Π, nor to the finite-n polygons, where $\bar{\Pi}$ consists of straight segments, but y_0 can not be neglected, as in [1, 12, 26].

for $0 \leq y \leq L/2$ and the opposite sign of the root for $L/2 \leq y \leq L$. $D(x,k) \equiv F(x,k) - E(x,k)$ here is the difference of elliptic integrals of the first and the second kinds respectively. Then,

$$L = 2y(C) = 2\sqrt{2}C\left(E - \frac{K}{2}\right) = 2\sqrt{2}C\frac{\pi}{4K}$$

where E and K are complete elliptic integrals of the first and the second kinds respectively taken at the value of elliptic modulus $k = k' = 1/\sqrt{2}$. Note that in this lemniscata point $K = \frac{\Gamma(1/4)^2}{4\sqrt{\pi}}$ and, using the Legendre formula

$$KE' + K'E - KK' = \frac{\pi}{2}$$

for the four complete elliptic integrals with complimentary modulus, one immediately obtains $E = \frac{\pi}{4K} + \frac{K}{2}$. Then, one obtains

$$L = \frac{\pi C}{\sqrt{2}K}, \quad \text{i.e.} \quad C = \frac{\sqrt{2}KL}{\pi}$$

The area is (μ^2 is the regulator)

$$S = 2L_\parallel C^2 \int_0^C \frac{dr}{(r^2 + \mu^2)\sqrt{C^4 - r^4}} = \frac{2L_\parallel}{C} \int_0^1 \frac{dr}{(r^2 + \mu^2)\sqrt{1 - r^4}}$$
$$= \frac{\sqrt{2}L_\parallel}{C} \frac{1}{1 + \mu^2} \Pi\left(-\frac{1}{1+\mu^2}, \frac{1}{\sqrt{2}}\right) \qquad \text{(C.7)}$$

where $\Pi(\nu, k)$ is the complete elliptic integral of the third kind. Its asymptotics can be found from the relation

$$k'^2 \frac{\sin\theta\cos\theta}{1 - k'^2\sin^2\theta}\left[\Pi\left(-(1 - k'^2\sin^2\theta), k\right) - K\right] = \frac{\pi}{2} - (E - K)F(\theta, k') - KE(\theta, k)$$

and using $F(\theta, k) = \theta + O(\theta^3)$, $E(\theta, k) = \theta + O(\theta^3)$:

$$\Pi\left(-\frac{1}{1+\mu^2}, \frac{1}{\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}\mu} - \frac{\pi}{2K} + O(\mu)$$

Then, the area

$$S = \frac{\pi L_\parallel}{C}\left(\frac{1}{\mu} - \frac{1}{\sqrt{2}K}\right) = \frac{\pi^2 L_\parallel}{\sqrt{2}KL}\left(\frac{1}{\mu} - \frac{1}{\sqrt{2}K}\right)$$

The finite piece in this answer is

$$S_{\text{fin}} = -\frac{\pi^2 L_\parallel}{2KL} = -\frac{(2\pi)^3}{\Gamma(1/4)^4}\frac{L_\parallel}{L} \qquad \text{(C.8)}$$

This result has to be compared with the double contour integral. Its finite part comes from the case when y and y' belong to two different parallel lines (when they belong to the same line one gets the contribution to the divergent term)

$$2L_\parallel \int_{-\infty}^{+\infty} \frac{d\xi}{\xi^2 + L^2} = 2\pi\frac{L_\parallel}{L} \qquad \text{(C.9)}$$

The difference between 2π in (C.9) and the coefficient in (C.8) is the confusing problem discovered in [23]. One can formulate our result as a non-trivial generalization of this statement:

  • A *similar* coefficient discrepancy exists for a circle of arbitrary shape, but only *few* of infinitely many coefficients are *different*.

D. MAPLE programs

We append here two simple MAPLE programs that one can use for evaluating the minimal area and the double contour integral (the latter one up to any given order in h). Using this program to obtain the area up to h^4 order and higher requires the knowledge of solution to the NG equation up to this order.

D.1 Calculation of A_Π

Literally, this program calculates the finite part $CCfin$ of the coefficient in front of the cubic term $h_k h_l \bar{h}_{k+l-1}$. It uses the explicit form (2.19) of the NG-harmonic functions.

```
>dH:=z->1+s*dh(z): dHH:=z->1+s*dhh(z):r2:=1-z*zz+s*a(z,zz):
>
> S:=sqrt( dH(z)*dHH(zz)*(dH(z)*dHH(zz)*r2 + diff(r2,z)*diff(r2,zz)) )/r2^(1/2)/(r2+mu^2);
>
> SS:=mtaylor(simplify(mtaylor(S,s,1)*(1-z*zz+mu^2)^(3/2)),c,2);
> SL:=simplify(simplify(mtaylor(S,s,2)-mtaylor(S,s,1))*(1-z*zz+mu^2)^(5/2)/s);
> SQ:=simplify(simplify(mtaylor(S,s,3)-mtaylor(S,s,2))*(1-z*zz+mu^2)^(7/2)/s^2):
> SC:=simplify(simplify(mtaylor(S,s,4)-mtaylor(S,s,3))*(1-z*zz+mu^2)^(9/2)/s^2):
>
> A:=(k,z,zz)->(1+(k-1)*sqrt(1-z*zz))*(1-sqrt(1-z*zz))^(k-1)/(z*zz)^(k-1)-z*zz;
> K:=5: L:=5: M:=K+L-1:
> h:=z->h[K]*z^K+h[L]*z^L; hh:=z->hh[M]*z^(M);
>
> dh:=z->diff(h(z),z); dhh:=z->diff(hh(z),z);
> a:=(z,zz)-> h[K]*z^(K-1)*A(K,z,zz) + h[L]*z^(L-1)*A(L,z,zz) + hh[M]*zz^(M-1)*A(M,z,zz) ;
> diff(a(z,zz),z):
> SS1:=simplify(SS);
> SL1:=simplify(SL);
> SQ1:=simplify(SQ);
> SC1:=simplify(SC);
>
> z:=sqrt(X)*exp(I*phi): zz:=sqrt(X)*exp(-I*phi):
> #simplify(SL1);
> SLI:=factor(int(simplify(SL1),phi=0..2*Pi)/2/Pi);
> SQI:=factor(int(simplify(SQ1),phi=0..2*Pi)/2/Pi);
> SCI:=factor(int(simplify(SC1),phi=0..2*Pi)/2/Pi);
>
> LL:=factor(int(SLI/((1-X+mu^2)^(5/2)),X=0..1));
> QQ:=factor(int(SQI/((1-X+mu^2)^(7/2)),X=0..1));
> CC:=factor(int(SCI/((1-X+mu^2)^(9/2)),X=0..1));
>
> QQQ:=coeff(QQ,arctan(1/mu)); QQQQ:=subs(mu=0,simplify(QQ-QQQ*arctan(1/mu)));
```

```
> QQdiv:=coeff(simplify(QQQ*mu*(1+mu^2)^7),mu,0);
> QQfin:=simplify(QQQQ-QQdiv);
>
> CCC:=coeff(CC,arctan(1/mu)); CCCC:=subs(mu=0,simplify(CC-CCC*arctan(1/mu)));
> CCdiv:=coeff(simplify(CCC*mu*(1+mu^2)^11),mu,0);
> CCfin:=simplify(CCCC-CCdiv);
```

D.2 Calculation of D_Π

N here denotes the number of switched on h_k, $NN \leq k \leq N$, and the calculation is performed with the accuracy $O(h^N)$.

```
> p:=3:
> NN:=0: N:=6:
>
> # theta=Phi,  phi = varphi
>
> U:=exp(2*I*phi) + t*sum( (k*h[k]*exp(I*(k-1)*theta) +
> k*hh[k]*exp(-I*(k-1)*theta))*exp(I*(k+1)*phi), k=NN..N) +
> t^2*sum(sum( k*l*h[k]*hh[l]*exp(I*(k-1)*theta)*exp(I*(k+1)*phi),
> l=NN..N),k=NN..N)  +  exp(-2*I*phi) +
> t*sum( (k*hh[k]*exp(-I*(k-1)*theta) +
> k*h[k]*exp(I*(k-1)*theta))*exp(-I*(k+1)*phi), k=NN..N) +
> t^2*sum(sum( k*l*hh[k]*h[l]*exp(-I*(k-1)*theta)*exp(-I*(k+1)*phi),
> l=NN..N),k=NN..N);
>
> V:= simplify(1 + t*sum( simplify(sin(k*phi)/sin(phi))*(h[k]*exp(I*(k-1)*theta)+
> hh[k]*exp(-I*(k-1)*theta)), k=NN..N) +
> t^2*sum(sum( simplify(sin(k*phi)/sin(phi)*sin(l*phi)/sin(phi))*(h[k]*hh[l]*
> exp(I*(k-l)*theta)+hh[k]*h[l]*exp(-I*(k-l)*theta))/2, l=NN..N), k=NN..N));
>
> Ra:=(mtaylor(U/2/(4*sin(phi)^2*V + lambda^2),t,p+1));
> RA:=simplify(int( Ra, theta = 0..2*Pi)/2/Pi-1/(4*sin(phi)^2) + 1/2);
>
> DI:=simplify(int( RA, phi=0..2*Pi )/2/Pi);
```

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